Lösungen zu Grenzwerten

Afg 1

1) $f(x) = 3x^4 - 7x + 18$

$$\lim_{x\to\infty} f(x) = \infty - \infty \quad \text{(unbestimmter Ausdruck)}$$

daher

$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} x^4\left(3 - \frac{7}{x^3} + \frac{18}{x^4}\right)$$

$$= \infty \cdot (3 - 0 + 0) = \infty \cdot 3 = \infty$$

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2) $f(x) = 2x^5 + 3x + 6$

$$\lim_{x\to\infty} f(x) = \infty + \infty = \infty$$

$$\lim_{x\to-\infty} f(x) = 2(-\infty)^5 + 3(-\infty) + 6 = -\infty - \infty$$
$$= -\infty$$

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3) $f(x) = -3x^4 - x^2$

$$\lim_{|x|\to\infty} f(x) = -\infty - \infty = -\infty$$

(2)

4) $f(x) = 4x^3 - x^2 + 3x + 2$

$$\lim_{x\to\infty} f(x) = \infty - \infty + \infty \quad \text{(unbestimmter Ausdruck)}$$

daher

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^3\left(4 - \frac{1}{x} + \frac{3}{x^2} + \frac{2}{x^3}\right)$$

$$= \infty(4-0+0+0) = \infty \cdot 4 = \infty$$

$$\lim_{x\to-\infty} f(x) = (-\infty)(4+0+0-0) = -\infty \cdot 4 = -\infty$$

Afg 2

1) $f(x) = \dfrac{x-1}{x-3}$ mit $D = \mathbb{R} \setminus \{3\}$

$$\lim_{x\to\infty} f(x) = \frac{\infty}{\infty} \quad \text{(unbestimmter Ausdruck)}$$

daher

$$\lim_{|x|\to\infty} f(x) \overset{:x}{\underset{:x}{=}} \lim_{|x|\to\infty} \frac{1-\frac{1}{x}}{1-\frac{3}{x}} = \frac{1-0}{1-0} = 1$$

$\Rightarrow$ horizontale Asymptote: $\boxed{y = 1}$

$$\left.\begin{aligned} \lim_{x\to3+} f(x) &= \frac{2}{0+} = \infty \\ \lim_{x\to3-} f(x) &= \frac{2}{0-} = -\infty \end{aligned}\right\} \text{vertikale Asymp:} \quad \boxed{x = 3}$$

2) $f(x) = \frac{2x+1}{x+2}$ mit $\mathbb{D} = \mathbb{R} \setminus \{-2\}$

$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} \frac{2+\frac{1}{x}}{1+\frac{2}{x}} = \frac{2+0}{1+0} = 2$$

$\Rightarrow$ horizontale Asymp: $\boxed{y = 2}$

$$\left.\begin{aligned} \lim_{x\to -2+} f(x) &= \frac{-3}{0+} = -\infty \\ \lim_{x\to -2-} f(x) &= \frac{-3}{0-} = \infty \end{aligned}\right\}$$ vertikale Asymp: $\boxed{x = -2}$

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3) $f(x) = \frac{3x^2+1}{x^2}$ mit $\mathbb{D} = \mathbb{R} \setminus \{0\}$

$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} \left(3 + \frac{1}{x^2}\right) = 3+0 = 3$$

$\Rightarrow$ horizontale Asymp: $\boxed{y = 3}$

$$\lim_{x\to 0} f(x) = \frac{1}{0+} = \infty \Rightarrow$$ vertikale Asymp: $\boxed{x = 0}$

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4) $f(x) = \frac{x^2}{x+1}$ mit $\mathbb{D} = \mathbb{R} \setminus \{-1\}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{1+\frac{1}{x}} = \frac{\infty}{1+0} = \infty$$

$$\lim_{x\to -\infty} f(x) = \frac{-\infty}{1-0} = -\infty$$

Ermittlung der schiefen Asymp:

$$x^2 : (x+1) = x - 1 + \frac{1}{x+1}$$

$$\begin{array}{l} x^2 : (x+1) \\ -(x^2+x) \\ \hline \quad -x \\ -(-x-1) \\ \hline \qquad 1 \end{array}$$

oder mit Horner-Schema:

	x^2	x	x^0
	1	0	0
-1	0	-1	1
Σ	1	-1	1
	x	x^0	$(x+1)^{-1}$

man sieht: Schiefe Asymp: $\boxed{y = x - 1}$

da $\lim\limits_{|x|\to\infty} \left(f(x) - (x-1) \right) = \lim\limits_{|x|\to\infty} \frac{1}{x+1} = 0$

Weiter:

$$\left.\begin{array}{l} \lim\limits_{x\to -1+} f(x) = \frac{1}{0+} = \infty \\ \lim\limits_{x\to -1-} f(x) = \frac{1}{0-} = -\infty \end{array}\right\} \text{ Vertikale Asymp.: } \boxed{x = -1}$$

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5) $f(x) = \dfrac{3x^2 - 4x + 1}{x-2}$ mit $\mathbb{D} = \mathbb{R} \setminus \{2\}$

zunächst zerlegen

	3	-4	1
2	0	6	4
Σ	3	2	5

somit $f(x) = 3x + 2 + \dfrac{5}{x-2}$

$\lim\limits_{x\to\infty} f(x) = \infty + 0 = \infty$; $\lim\limits_{x\to -\infty} f(x) = -\infty + 0 = -\infty$

Schiefe Asymp: $\boxed{y = 3x + 2}$

Alternative:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x-4+\frac{1}{x}}{1-\frac{2}{x}} = \frac{\infty+0}{1-0} = \infty$$

$$\lim_{x\to-\infty} f(x) = \frac{-\infty-0}{1+0} = -\infty$$

Weiter:

$$\lim_{x\to 2+} f(x) = \lim_{x\to 2+} \left(3x+2+\frac{5}{x-2}\right) = 8+\frac{5}{0+} = \infty$$

$$\lim_{x\to 2-} f(x) = 8+\frac{5}{0-} = -\infty$$

$\Rightarrow$ vertikale Asymp: $\boxed{x=2}$

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6) $f(x) = \dfrac{x^3-1}{x^2+1}$ mit $\mathbb{D} = \mathbb{R}$

zuerst zerlegen: $\quad\to$ schiefe Asymp: $\boxed{y=x}$

$$(x^3-1) : (x^2+1) = x - \frac{x+1}{x^2+1}$$
$$\underline{-(x^3+x)}$$
$$-x-1$$

d.h. $f(x) = x - \dfrac{x+1}{x^2+1}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{\infty-0}{1+0} = \infty$$

$$\lim_{x\to-\infty} f(x) = \frac{-\infty-0}{1+0} = -\infty$$

7) $f(x) = \frac{4x^2-1}{(x+3)^2}$ mit $\mathbb{D} = \mathbb{R} \setminus \{-3\}$ 6

$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} \frac{4x^2-1}{x^2+6x+9} = \lim_{|x|\to\infty} \frac{4-\frac{1}{x^2}}{1+\frac{6}{x}+\frac{9}{x^2}}$$

$$= \frac{4-0}{1+0+0} = 4$$

$\Longrightarrow$ horizontale Asymp: $\boxed{y = 4}$

$$\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{4x^2-1}{(x+3)^2} = \frac{35}{0+} = \infty$$

$\Longrightarrow$ vertikale Asymp: $\boxed{x = -3}$

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8) $f(x) = \frac{5x}{x^2-4}$ mit $\mathbb{D} = \mathbb{R} \setminus \{-2; 2\}$

$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} \frac{5}{x-\frac{4}{x}} = \frac{5}{\pm\infty \pm 0} = 0$$

$\Longrightarrow$ horizontale Asymp: $\boxed{y = 0}$

$$\lim_{x\to 2+} f(x) = \frac{10}{0+} = \infty$$

$$\lim_{x\to 2-} f(x) = \frac{10}{0-} = -\infty$$

$$\lim_{x\to -2+} f(x) = \frac{-10}{0-} = \infty$$

$$\lim_{x\to -2-} f(x) = \frac{-10}{0+} = -\infty$$

vertikale Asymp:

$\boxed{x = 2}$

$\boxed{x = -2}$

Alternative: Sei jeweils $h > 0$:

$$\lim_{x \to 2+} \frac{5x}{x^2-4} = \lim_{h \to 0} \frac{5(2+h)}{(2+h)^2-4} = \lim_{h \to 0} \frac{5(2+h)}{4+4h+h^2-4}$$

$$= \lim_{h \to 0} \frac{5(2+h)}{4h+h^2} = \frac{10}{0+} = \infty$$

$$\lim_{x \to 2-} \frac{5x}{x^2-4} = \lim_{h \to 0} \frac{5(2-h)}{(2-h)^2-4} = \lim_{h \to 0} \frac{5(2-h)}{4-4h+h^2-4}$$

$$= \lim_{h \to 0} \frac{5(2-h)}{h^2-4h} = \lim_{h \to 0} \frac{5(2-h)}{h(h-4)}$$

$$= \frac{10}{(0+)\cdot(-4)} = \frac{10}{0-} = -\infty$$

$$\lim_{x \to -2+} \frac{5x}{x^2-4} = \lim_{h \to 0} \frac{5(-2+h)}{(-2+h)^2-4} = \lim_{h \to 0} \frac{5(-2+h)}{4-4h+h^2-4}$$

$$= \lim_{h \to 0} \frac{5(-2+h)}{h^2-4h} = \lim_{h \to 0} \frac{5(-2+h)}{h(h-4)}$$

$$= \frac{-10}{(0+)(-4)} = \frac{-10}{0-} = \infty$$

$$\lim_{x \to -2-} \frac{5x}{x^2-4} = \lim_{h \to 0} \frac{5(-2-h)}{(-2-h)^2-4} = \lim_{h \to 0} \frac{5(-2-h)}{4+4h+h^2-4}$$

$$= \lim_{h \to 0} \frac{5(-2-h)}{4h+h^2} = \frac{-10}{0+} = -\infty$$

9) $f(x) = \frac{1}{x} - \frac{1}{x^2}$ mit $\mathbb{D} = \mathbb{R} \setminus \{0\}$

$\lim_{|x| \to \infty} f(x) = 0 - 0 = 0$

$\Rightarrow$ horizontale Asymp: $\boxed{y = 0}$

$\lim_{x \to 0+} f(x) = \frac{1}{0+} - \frac{1}{0+} = \infty - \infty$

(unbestimmter Ausdruck)

daher Termumformung:

$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x-1}{x^2} = \frac{-1}{0+} = -\infty$

$\left[\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \left(\frac{1}{x} - \frac{1}{x^2} \right) = \frac{1}{0-} - \frac{1}{0+} = -\infty - \infty = -\infty \right]$

wäre auch gegangen!

$\Rightarrow$ vertikale Asymp: $\boxed{x = 0}$

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10) $f(x) = \frac{2x}{x^2-1} - \frac{x+1}{x-1}$ mit $\mathbb{D} = \mathbb{R} \setminus \{-1; 1\}$

am besten erst auf einen Bruchstrich bringen:

$f(x) = \frac{2x - (x+1)^2}{x^2-1} = \frac{2x - x^2 - 2x - 1}{x^2 - 1}$

$\boxed{f(x) = \frac{-x^2 - 1}{x^2 - 1}}$

$\longrightarrow$

$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} \frac{-1-\frac{1}{x^2}}{1-\frac{1}{x^2}} = \frac{-1-0}{1-0} = -1$$

$\Longrightarrow$ horizontale Asymp: $\boxed{y=-1}$

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} \frac{-x^2-1}{x^2-1} = \frac{-2}{0+} = -\infty$$

$$\lim_{x\to 1-} f(x) = \frac{-2}{0-} = \infty$$

$$\lim_{x\to -1+} f(x) = \frac{-2}{0-} = \infty$$

$$\lim_{x\to -1-} f(x) = \frac{-2}{0+} = -\infty$$

vertikale Asymptoten: $\boxed{x=1}$ $\boxed{x=-1}$

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11) $f(x) = \dfrac{x^2+x-2}{(x-1)^2}$ mit $\mathbb{D} = \mathbb{R}\setminus\{1\}$

$\lim\limits_{x\to 1} f(x) = \dfrac{0}{0+}$ (unbestimmter Ausdruck)

daher Termumformung vornehmen:

$$\frac{x^2+x-2}{(x-1)^2} = \frac{(x-1)(x+2)}{(x-1)^2} = \frac{x+2}{x-1}$$

somit $\lim\limits_{x\to 1+} f(x) = \dfrac{3}{0+} = \infty$

$\lim\limits_{x\to 1-} f(x) = \dfrac{3}{0-} = -\infty$

vertikale Asymp: $\boxed{x=1}$

$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} \frac{x+2}{x-1} = \lim_{|x|\to\infty} \frac{1+\frac{2}{x}}{1-\frac{1}{x}} = 1$$

$\Rightarrow$ horizontale Asymp: $\boxed{y = 1}$

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12) $f(x) = \dfrac{x^2-2x+1}{x(x-1)}$ mit $\mathbb{D} = \mathbb{R}\setminus\{0;1\}$

$$\lim_{x\to 1+} f(x) = \frac{0}{0+} \quad (\text{unbestimmter Ausdruck})$$

daher Termumformung vornehmen:

$$\frac{x^2-2x+1}{x(x-1)} = \frac{(x-1)^2}{x(x-1)} = \frac{x-1}{x} = 1-\frac{1}{x}$$

$$\Rightarrow \lim_{x\to 1} f(x) = \frac{0}{1} = 0 \quad \left(\text{d.h. } x=1 \text{ ist hebbare Def. lücke}\right)$$

$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} \frac{x-1}{x} = \lim_{|x|\to\infty}\left(1-\frac{1}{x}\right) = 1$$

$\Rightarrow$ horizontale Asymp: $\boxed{y = 1}$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{x-1}{x} = \frac{-1}{0+} = -\infty$$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{x-1}{x} = \frac{-1}{0-} = \infty$$

$\Rightarrow$ vertikale Asymp: $\boxed{x = 0}$