Lösungen zum Übungsblatt: "Ableitungsregeln"

87) $f(x) = \frac{x^2}{3} \Rightarrow f'(x) = \frac{2x}{3}$; $t(x) = mx + c$

$m = f'(3) = 2 \Rightarrow t(x) = 2x + c$

$t(3) = f(3) = 3 \Rightarrow 2 \cdot 3 + c = 3 \Rightarrow c = -3$

$\Rightarrow \boxed{t(x) = 2x - 3}$ Tangente $\Rightarrow \boxed{h = t(6) = 9}$

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88) Quotientenregel

a) $f(x) = \frac{1}{a+bx} \Rightarrow f'(x) = \frac{0 - 1 \cdot b}{(a+bx)^2} = \boxed{-\frac{b}{(a+bx)^2}}$

b) $f(x) = \frac{1}{1+x} \Rightarrow f'(x) = \frac{0 - 1 \cdot 1}{(1+x)^2} = \boxed{-\frac{1}{(1+x)^2}}$

c) $f(x) = \frac{2x^3}{a^2 - x^2} \Rightarrow f'(x) = \frac{6x^2(a^2 - x^2) - 2x^3(-2x)}{(a^2 - x^2)^2}$

$= \frac{6a^2x^2 - 6x^4 + 4x^4}{(a^2 - x^2)^2} = \boxed{\frac{6a^2x^2 - 2x^4}{(a^2 - x^2)^2}}$

d) $f(x) = \frac{1 - x^3}{1 + x^3} \Rightarrow f'(x) = \frac{-3x^2(1+x^3) - (1-x^3) \cdot 3x^2}{(1+x^3)^2}$

$= \frac{-3x^2 - 3x^5 - 3x^2 + 3x^5}{(1+x^3)^2} = \boxed{\frac{-6x^2}{(1+x^3)^2}}$

e) $f(x) = \frac{3x^3 + 2}{x^2 + 1} \Rightarrow f'(x) = \frac{9x^2(x^2+1) - (3x^3+2) \cdot 2x}{(x^2+1)^2}$

$= \frac{9x^4 + 9x^2 - 6x^4 - 4x}{(x^2+1)^2} = \boxed{\frac{3x^4 + 9x^2 - 4x}{(x^2+1)^2}}$

f) $f(x) = \frac{2x^2 - 1}{x^2 - 2} \Rightarrow f'(x) = \frac{4x(x^2 - 2) - (2x^2 - 1) \cdot 2x}{(x^2 - 2)^2}$

$= \frac{4x^3 - 8x - 4x^3 + 2x}{(x^2 - 2)^2} = \boxed{\frac{-6x}{(x^2 - 2)^2}}$

g) $f(x) = \dfrac{x^6+4x^3-3}{x^6-4x^3+3}$

$$\Rightarrow f'(x) = \frac{(6x^5+12x^2)(x^6-4x^3+3)-(x^6+4x^3-3)(6x^5-12x^2)}{(x^6-4x^3+3)^2}$$

$$= \frac{\cancel{6x^{11}}-12x^8-30x^5+\cancel{36x^2}-\cancel{6x^{11}}-12x^8+66x^5-\cancel{36x^2}}{(x^6-4x^3+3)^2}$$

$$\Rightarrow \boxed{f'(x) = \frac{36x^5-24x^8}{(x^6-4x^3+3)^2}}$$

h) $f(x) = \dfrac{x^2-2x+4}{x^2+2x-4}$

$$f'(x) = \frac{(2x-2)(x^2+2x-4)-(x^2-2x+4)(2x+2)}{(x^2+2x-4)^2}$$

$$= \frac{\cancel{2x^3}+2x^2-12x+\cancel{8}-\cancel{2x^3}+2x^2-4x-\cancel{8}}{(x^2+2x-4)^2} = \boxed{\frac{4x^2-16x}{(x^2+2x-4)^2}}$$

i) $f(x) = \dfrac{1-3x^3}{1+3x^3} \Rightarrow f'(x) = \dfrac{-9x^2(1+3x^3)-(1-3x^3)\cdot 9x^2}{(1+3x^3)^2}$

$$= \frac{-9x^2-\cancel{27x^5}-9x^2+\cancel{27x^5}}{(1+3x^3)^2} = \boxed{\frac{-18x^2}{(1+3x^3)^2}}$$

k) $f(x) = \dfrac{3x^2+1}{1-x^2} \Rightarrow f'(x) = \dfrac{6x(1-x^2)-(3x^2+1)(-2x)}{(1-x^2)^2}$

$$= \frac{6x-\cancel{6x^3}+\cancel{6x^3}+2x}{(1-x^2)^2} = \boxed{\frac{8x}{(1-x^2)^2}}$$

l) $f(x) = \dfrac{7x+9}{11x+5} \Rightarrow f'(x) = \dfrac{7(11x+5)-11(7x+9)}{(11x+5)^2}$

$$= \frac{\cancel{77x}+35-\cancel{77x}-99}{(11x+5)^2} = \boxed{\frac{-64}{(11x+5)^2}}$$

zu 88)

m) $f(x) = \frac{a+x}{b+x} \Rightarrow f'(x) = \frac{1\cdot(b+x) - (a+x)\cdot 1}{(b+x)^2}$

$= \frac{b+x-a-x}{(b+x)^2} = \boxed{\frac{b-a}{(b+x)^2}}$

n) $f(x) = \frac{1}{x^2} + \frac{1}{x^3} = x^{-2} + x^{-3}$

$\Rightarrow f'(x) = -2x^{-3} - 3x^{-4} = \boxed{-\frac{2}{x^3} - \frac{3}{x^4}}$

o) $f(x) = \frac{2}{x^2} + 1 = 2x^{-2} + 1$

$\Rightarrow f'(x) = -4x^{-3} = \boxed{-\frac{4}{x^3}}$

p) $f(x) = \frac{x^2-4}{\sqrt{x}} \Rightarrow f'(x) = \frac{2x\cdot\sqrt{x} - (x^2-4)\cdot\frac{1}{2\sqrt{x}}}{x} \quad \begin{matrix}\cdot 2\sqrt{x} \\ \cdot 2\sqrt{x}\end{matrix}$

$= \frac{4x^2 - (x^2-4)}{2x\sqrt{x}} = \frac{4x^2 - x^2 + 4}{2\cdot x^{3/2}} = \boxed{\frac{3x^2+4}{2\cdot\sqrt{x^3}}}$

q) $f(x) = \frac{1}{1+\sqrt[4]{x}}$; $(\sqrt[4]{x})' = (x^{\frac{1}{4}})' = \frac{1}{4}\cdot x^{-\frac{3}{4}} = \frac{1}{4\cdot\sqrt[4]{x^3}}$

$\Rightarrow f'(x) = \frac{0 - 1\cdot\frac{1}{4\cdot\sqrt[4]{x^3}}}{(1+\sqrt[4]{x})^2} = \boxed{\frac{-1}{4(1+\sqrt[4]{x})^2\cdot\sqrt[4]{x^3}}}$

r) $f(x) = \frac{1+\sqrt{x}}{1-\sqrt{x}} \Rightarrow f'(x) = \frac{\frac{1}{2\sqrt{x}}(1-\sqrt{x}) - (1+\sqrt{x})(-\frac{1}{2\sqrt{x}})}{(1-\sqrt{x})^2}$

$= \frac{\frac{1}{2\sqrt{x}} \cancel{-\frac{1}{2}} + \frac{1}{2\sqrt{x}} \cancel{+\frac{1}{2}}}{(1-\sqrt{x})^2} = \boxed{\frac{1}{\sqrt{x}(1-\sqrt{x})^2}}$

s) $f(x) = \frac{a+\sqrt{x}}{a-\sqrt{x}} \Rightarrow f'(x) = \frac{\frac{1}{2\sqrt{x}}(a-\sqrt{x}) - (a+\sqrt{x})(-\frac{1}{2\sqrt{x}})}{(a-\sqrt{x})^2}$

$= \frac{\frac{a}{2\sqrt{x}} \cancel{-\frac{1}{2}} + \frac{a}{2\sqrt{x}} \cancel{+\frac{1}{2}}}{(a-\sqrt{x})^2} = \boxed{\frac{a}{\sqrt{x}(a-\sqrt{x})^2}}$

Zu 88)

t) $f(x) = \frac{a^2 - x^2}{\sqrt{x}\,(x-a)} = \frac{(a+x)\cancel{(a-x)}}{-\sqrt{x}\,\cancel{(a-x)}} = -\frac{a+x}{\sqrt{x}}$

$$\Rightarrow f'(x) = -\frac{1\cdot\sqrt{x} - (a+x)\cdot\frac{1}{2\sqrt{x}}}{x} \quad \begin{matrix}\cdot 2\sqrt{x}\\ \cdot 2\sqrt{x}\end{matrix}$$

$$= -\frac{2x - a - x}{2x\sqrt{x}} = \boxed{\frac{a-x}{2\sqrt{x^3}}}$$

u) $f(x) = 2x^3 + \frac{1}{1-x}$

$$f'(x) = 6x^2 + \frac{0 - 1\cdot(-1)}{(1-x)^2} = \boxed{6x^2 + \frac{1}{(1-x)^2}}$$

v) $f(x) = a^2x^3 + 2x + \frac{2}{1-x^2}$

$$f'(x) = 3a^2x^2 + 2 + \frac{0 - 2\cdot(-2x)}{(1-x^2)^2} = \boxed{3a^2x^2 + 2 + \frac{4x}{(1-x^2)^2}}$$

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92) $g_a(x) = \frac{ax^3+9}{1-x^2}$; $a \in \mathbb{R}$; $\mathbb{D} = \mathbb{R}\setminus\{\pm 1\}$

$$g_a'(x) = \frac{3ax^2(1-x^2) - (ax^3+9)(-2x)}{(1-x^2)^2}$$

$$= \frac{3ax^2 - 3ax^4 + 2ax^4 + 18x}{(1-x^2)^2} = \boxed{\frac{3ax^2 - ax^4 + 18x}{(1-x^2)^2}}$$

a) Sei $a = 1 \Rightarrow g_1'(x) = \frac{3x^2 - x^4 + 18x}{(1-x^2)^2} \Rightarrow g_1'(-3) = -\frac{27}{16}$

$\Rightarrow t(x) = -\frac{27}{16}x + c$; $t(-3) = g_1(-3) = \frac{9}{4}$

$\Rightarrow -\frac{27}{16}\cdot(-3) + c = \frac{9}{4} \Rightarrow c = -\frac{45}{16} \Rightarrow \boxed{t(x) = -\frac{27}{16}x - \frac{45}{16}}$

zu 92)

b) $$g_a'(2) = \frac{12a - 16a + 36}{(-3)^2} = \boxed{\frac{36-4a}{9}}$$

$$g_a'(2) = 0 \iff 36 - 4a = 0 \iff \boxed{a = 9}$$

c) $$g_a'(2) = \frac{8}{3} \iff \frac{36-4a}{9} = \frac{8}{3} \iff 36 - 4a = 24$$
$$\iff 4a = 12$$
$$\iff \boxed{a = 3}$$

d) $$t(x) = mx + c \quad ; \quad m = \frac{36-4a}{9} \quad ; \quad g_a(2) = \frac{8a+9}{-3}$$

$$g_a(2) = t(2) = 2m + c$$

$$\Rightarrow c = g_a(2) - 2m = \frac{8a+9}{-3} - \frac{2(36-4a)}{9}$$
$$= \frac{-3(8a+9) - 2(36-4a)}{9} = \frac{-24a - 27 - 72 + 8a}{9} = \frac{-16a-99}{9}$$

t geht durch Ursprung $\iff c = 0 \iff -16a - 99 = 0$
$$\iff \boxed{a = -\frac{99}{16}}$$

93) Produktregel

a) $$f(x) = (3x^2+4)(2x^3-1) \Rightarrow f'(x) = 6x(2x^3-1) + (3x^2+4)\cdot 6x^2$$
$$= 12x^4 - 6x + 18x^4 + 24x^2 = \boxed{30x^4 + 24x^2 - 6x}$$

b) $$f(x) = (x^4+1)(x^4-1) \Rightarrow f'(x) = 4x^3(x^4-1) + (x^4+1)\cdot 4x^3$$
$$= 4x^7 - \cancel{4x^3} + 4x^7 + \cancel{4x^3} = \boxed{8x^7}$$

einfacher so: $$f(x) = (x^4)^2 - 1^2 = x^8 - 1$$
$$\Rightarrow \boxed{f'(x) = 8x^7}$$

c) $f(x) = (a^2+x^2)(a^2-x^2) = a^4 - x^4$

$\Rightarrow \boxed{f'(x) = -4x^3}$

d) $f(x) = (x^4+a)(4x^3+b)$

$\Rightarrow f'(x) = 4x^3(4x^3+b) + (x^4+a)\cdot 12x^2$

$= 16x^6 + 4bx^3 + 12x^6 + 12ax^2$

$\boxed{f'(x) = 28x^6 + 4bx^3 + 12ax^2}$

e) $f(x) = (ax+b)(b-ax) = b^2 - a^2x^2$

$\Rightarrow \boxed{f'(x) = -2a^2x}$

f) $f(x) = (\sqrt{x} + a)(x^2+2ax)$

$f'(x) = \frac{1}{2\sqrt{x}}(x^2+2ax) + (\sqrt{x}+a)(2x+2a)$

$= \frac{x^2}{2\sqrt{x}} + \frac{ax}{\sqrt{x}} + 2x\sqrt{x} + 2a\sqrt{x} + 2ax + 2a^2$

$= 0{,}5\,x\sqrt{x} + a\sqrt{x} + 2x\sqrt{x} + 2a\sqrt{x} + 2ax + 2a^2$

$= \boxed{2{,}5\,x\sqrt{x} + 3a\sqrt{x} + 2ax + 2a^2}$

g) $f(x) = (x+b)^2 = x^2+2bx+b^2 \Rightarrow \boxed{f'(x) = 2x+2b}$

h) $f(x) = (3x+2x^2+1)^2 = (3x+2x^2+1)(3x+2x^2+1)$

$f'(x) = 2(3+4x)(3x+2x^2+1) = (6+8x)(3x+2x^2+1)$

$= 18x + 12x^2 + 6 + 24x^2 + 16x^3 + 8x$

$= \boxed{16x^3 + 36x^2 + 26x + 6}$

94) Kettenregel und Gemischte Aufgaben

a) $f(x) = (x^2-10)^2 \Rightarrow f'(x) = 2(x^2-10)\cdot 2x = \boxed{4x(x^2-10)}$ $\boxed{4x^3-40x}$

b) $f(x) = (5-3x)^4 \Rightarrow f'(x) = 4(5-3x)^3\cdot(-3) = \boxed{-12(5-3x)^3}$

c) $f(x) = \left(x-\frac{1}{x}\right)^2 \Rightarrow f'(x) = 2\left(x-\frac{1}{x}\right)\cdot\left(1+\frac{1}{x^2}\right) = \boxed{2\left(x-\frac{1}{x^3}\right)}$

d) $f(x) = \left(x^2+\frac{1}{x}\right)^3 \Rightarrow f'(x) = 3\left(x^2+\frac{1}{x}\right)^2\left(2x-\frac{1}{x^2}\right) = \boxed{6x^5+9x^2-\frac{3}{x^4}}$

e) $f(x) = (x^2+x^3)^4 \Rightarrow f'(x) = 4(x^2+x^3)^3(2x+3x^2)$.

f) $f(x) = \frac{b}{a}\left(x^2+\frac{1}{x^3}\right)^3 \Rightarrow f'(x) = \frac{3b}{a}\left(x^2+\frac{1}{x^3}\right)^2\cdot\left(2x-\frac{3}{x^4}\right)$

$$\boxed{f'(x) = \frac{3b}{a}\left(2x^5-\frac{4}{x^5}-\frac{3}{x^{10}}+1\right)}$$

g) $f(x) = \frac{2x^3+1}{(3-2x^3)^4} \Rightarrow f'(x) = \frac{6x^2(3-2x^3)^4-(2x^3+1)\cdot 4(3-2x^3)^3(-6x^2)}{(3-2x^3)^8}$

$$= \frac{\cancel{(3-2x^3)^3}\left[6x^2(3-2x^3)+24x^2(2x^3+1)\right]}{(3-2x^3)^{\cancel{8}\,5}} = \boxed{\frac{36x^5+42x^2}{(3-2x^3)^5}}$$

h) $f(x) = \frac{3-x^2}{(4-2x)^3} \Rightarrow f'(x) = \frac{-2x(4-2x)^3-(3-x^2)\cdot 3(4-2x)^2\cdot(-2)}{(4-2x)^6}$

$$= \frac{\cancel{(4-2x)^2}\left[-2x(4-2x)+6(3-x^2)\right]}{(4-2x)^{\cancel{6}\,4}} = \boxed{\frac{18-2x^2-8x}{(4-2x)^4}}$$

$\longrightarrow$

zu 94)

i) $f(x) = x + 1 + \frac{(x^2+2)^2}{2-x^3}$

$$\Rightarrow f'(x) = 1 + \frac{2(x^2+2)\cdot 2x(2-x^3) - (x^2+2)^2\cdot(-3x^2)}{(2-x^3)^2}$$

$$= 1 + \frac{(x^2+2)\left[4x(2-x^3) + 3x^2(x^2+2)\right]}{(2-x^3)^2}$$

$$= 1 + \frac{(x^2+2)\left[8x - 4x^4 + 3x^4 + 6x^2\right]}{(2-x^3)^2}$$

$$\boxed{f'(x) = 1 + \frac{(x^2+2)\left[6x^2 - x^4 + 8x\right]}{(2-x^3)^2}} = \boxed{\frac{-x^6 + 4x^4 + 8x^3 + 12x^2 + 16x}{(2-x^3)^2}}$$

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k) $f(x) = 2x + \frac{1-x^2}{(2-x)^2}$

$$f'(x) = 2 + \frac{-2x(2-x)^2 - (1-x^2)\cdot 2(2-x)\cdot(-1)}{(2-x)^4}$$

$$= 2 + \frac{\cancel{(2-x)}\left[-2x(2-x) + 2(1-x^2)\right]}{(2-x)^{\cancel{4}3}}$$

$$= 2 + \frac{-4x + \cancel{2x^2} + 2 - \cancel{2x^2}}{(2-x)^3} = \boxed{2 + \frac{2-4x}{(2-x)^3}}$$