Lösungen zum Blatt 9

Afg 1

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}; \quad \det(A - \lambda E_2) = \begin{vmatrix} 1-\lambda & -1 \\ 0 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)$$

λ EW von $A \iff (1-\lambda)(2-\lambda) = 0$

$\boxed{\lambda_1 = 1}$; $\boxed{\lambda_2 = 2}$ EW von A

EV von A zu $\lambda = 1$:

$$(A - E_2)\vec{x} = \vec{0} \iff \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \left.\begin{array}{ll} \text{I} & 0x_1 - x_2 = 0 \\ \text{II} & 0x_1 + x_2 = 0 \end{array}\right\} \iff \boxed{x_2 = 0}$$

x_1 frei wählbar, $\boxed{x_1 = \alpha}$

$$\Rightarrow \mathrm{Eig}(A;1) = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$

EV sind von der Form $\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ mit $\alpha \neq 0$

EV von A zu $\lambda = 2$:

$$(A - 2E_2)\vec{x} = \vec{0} \iff \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff x_1 + x_2 = 0$$

wähle $\boxed{x_2 = \alpha} \Rightarrow \boxed{x_1 = -\alpha}$

$$\rightarrow \mathrm{Eig}(A;2) = \left\{ \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$

$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\det(B-\lambda E_2) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$

$\boxed{\lambda_1 = -1}$; $\boxed{\lambda_2 = 1}$ EW von B

EV von B zu $\lambda = -1$:

$(B + E_2)\vec{x} = \vec{0} \iff \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\iff x_1 + x_2 = 0$

wähle $\boxed{x_2 = \alpha} \implies \boxed{x_1 = -\alpha}$

$\implies Eig(B; -1) = \left\{ \alpha \begin{pmatrix} -1 \\ 1 \end{pmatrix} / \alpha \in \mathbb{R} \right\}$

EV von B zu $\lambda = 1$:

$(B - E_2)\vec{x} = \vec{0} \iff \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\iff x_1 - x_2 = 0$

wähle $\boxed{x_2 = \alpha} \implies \boxed{x_1 = \alpha}$

$\implies Eig(B; 1) = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \alpha \in \mathbb{R} \right\}$

$C = \begin{pmatrix} 5 & 1 \\ 4 & 2 \end{pmatrix}$; $\det(C - \lambda E_2) = \begin{vmatrix} 5-\lambda & 1 \\ 4 & 2-\lambda \end{vmatrix}$

$= (5-\lambda)(2-\lambda) - 4$

$= \lambda^2 - 7\lambda + 6$

$0 = (\lambda - 1)(\lambda - 6)$

$\boxed{\lambda_1 = 1}$; $\boxed{\lambda_2 = 6}$
EW von C

EV von C zu $\lambda = 1$:

$(C - E_2)\vec{x} = \vec{0} \iff \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\iff 4x_1 + x_2 = 0$

wähle $\boxed{x_1 = \alpha} \implies \boxed{x_2 = -4\alpha} \implies Eig(C; 1) = \left\{ \alpha \begin{pmatrix} 1 \\ -4 \end{pmatrix} / \alpha \in \mathbb{R} \right\}$

EV van C zu $\lambda = 6$:

$$(C - 6E_2)\vec{x} = \vec{o} \iff \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\iff x_1 - x_2 = 0$$

wähle $\boxed{x_2 = \alpha} \implies \boxed{x_1 = \alpha}$

$$\implies Eig(C;6) = \left\{\alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| \alpha \in \mathbb{R}\right\}.$$

Afg 2

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}$$

$$\det(A - \lambda E_3) = \begin{vmatrix} 2-\lambda & 2 & 3 \\ 1 & 2-\lambda & 1 \\ 2 & -2 & 1-\lambda \end{vmatrix}$$
$$= (2-\lambda)^2(1-\lambda) + 4 - 6 - 6(2-\lambda) - 2(1-\lambda) + 2(2-\lambda)$$
$$= (2-\lambda)^2(1-\lambda) + 6\lambda - 12$$
$$= (2-\lambda)^2(1-\lambda) - 6(\lambda - 2)$$
$$= (2-\lambda)\left[(2-\lambda)(1-\lambda) - 6\right]$$
$$= (2-\lambda)(\lambda^2 - 3\lambda - 4)$$
$$0 = (2-\lambda)(\lambda+1)(\lambda-4)$$

$\boxed{\lambda_1 = -1}$; $\boxed{\lambda_2 = 2}$; $\boxed{\lambda_3 = 4}$ EW van A

EV van A zu $\lambda = -1$:

$$A + E_3 = \begin{pmatrix} 3 & 2 & 3 \\ 1 & 3 & 1 \\ 2 & -2 & 2 \end{pmatrix} \xrightarrow{I + III} \begin{pmatrix} 3 & 2 & 3 \\ 1 & 3 & 1 \\ 5 & 0 & 5 \end{pmatrix}$$

III $5x_1 + 5x_3 = 0 \iff x_1 + x_3 = 0$

wähle $\boxed{x_3 = \alpha} \implies \boxed{x_1 = -\alpha}$

I $3x_1 + 2x_2 + 3x_3 = 0 \implies -3\alpha + 2x_2 + 3\alpha = 0$

$\iff \boxed{x_2 = 0}$

$$\implies Eig(A;-1) = \left\{\alpha\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \middle| \alpha \in \mathbb{R}\right\}$$

EV von A zu $\lambda = 2$:

$$A - 2E_3 = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & -2 & -1 \end{pmatrix}$$

I: $2x_2 + 3x_3 = 0$

wähle $\boxed{x_3 = 2\alpha} \Rightarrow \boxed{x_2 = -3\alpha}$

II $x_1 + x_3 = 0 \Rightarrow \boxed{x_1 = -2\alpha}$

$$\Rightarrow Eig(A;2) = \left\{ \alpha \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$

oder auch $Eig(A;2) = \left\{ \alpha \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$

EV von A zu $\lambda = 4$:

$$A - 4E_3 = \begin{pmatrix} -2 & 2 & 3 \\ 1 & -2 & 1 \\ 2 & -2 & -3 \end{pmatrix} \xrightarrow{I+II} \begin{pmatrix} -2 & 2 & 3 \\ -1 & 0 & 4 \\ 2 & -2 & -3 \end{pmatrix}$$

II: $-x_1 + 4x_3 = 0$; wähle $\boxed{x_3 = \alpha} \Rightarrow \boxed{x_1 = 4\alpha}$

III: $2x_1 - 2x_2 - 3x_3 = 0 \Rightarrow 2x_2 = 2x_1 - 3x_3$

$$= 8\alpha - 3\alpha = 5\alpha$$

$$\Rightarrow \boxed{x_2 = 2{,}5\alpha}$$

$$\Rightarrow Eig(A;4) = \left\{ \alpha \begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$

$$B = \begin{pmatrix} -5 & 0 & 7 \\ 6 & 2 & -6 \\ -4 & 0 & 6 \end{pmatrix}$$

$$\det(B - \lambda E_3) = \begin{vmatrix} -5-\lambda & 0 & 7 \\ 6 & 2-\lambda & -6 \\ -4 & 0 & 6-\lambda \end{vmatrix}$$

$$= (-5-\lambda)(2-\lambda)(6-\lambda) + 28(2-\lambda)$$

$$= (2-\lambda)\left[(-5-\lambda)(6-\lambda) + 28\right]$$

$$= (2-\lambda)(\lambda^2 - \lambda - 2)$$

$$= (2-\lambda)(\lambda+1)(\lambda-2)$$

$$0 = -(\lambda-2)^2(\lambda+1)$$

$\boxed{\lambda_1 = -1}$; $\boxed{\lambda_2 = 2}$ EW von B

EV von B zu $\lambda = -1$:

$$B + E_3 = \begin{pmatrix} -4 & 0 & 7 \\ 6 & 3 & -6 \\ -4 & 0 & 7 \end{pmatrix} \quad \text{I}: -4x_1 + 7x_3 = 0$$

wähle $\boxed{x_3 = 4\alpha}$

$2x_1 + x_2 - 2x_3 = 0 \quad \longrightarrow \boxed{x_1 = 7\alpha}$

$\text{II} \quad 6x_1 + 3x_2 - 6x_3 = 0$

$\Longrightarrow \boxed{x_2 = -6\alpha}$

$$\Longrightarrow \operatorname{Eig}(B; -1) = \left\{ \alpha \begin{pmatrix} 7 \\ -6 \\ 4 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$

EV von B zu $\lambda = 2$:

$$B - 2E_3 = \begin{pmatrix} -7 & 0 & 7 \\ 6 & 0 & -6 \\ -4 & 0 & 4 \end{pmatrix} \Longleftrightarrow x_1 - x_3 = 0 \Longleftrightarrow x_1 = x_3$$

wähle $\boxed{x_1 = x_3 = \alpha}$ und $\boxed{x_2 = \beta}$

$$\operatorname{Eig}(B; 2) = \left\{ \begin{pmatrix} \alpha \\ \beta \\ \alpha \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}$$

$$\begin{pmatrix} \alpha \\ \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ \alpha \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

z. B. sind $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ und $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ EV von B zum EW $\lambda = 2$

$$\Longrightarrow \operatorname{Eig}(B; 2) = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}$$

Afg 3

$$A = \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

$$\det(A - \lambda E_3) = \begin{vmatrix} 7-\lambda & 2 & 0 \\ 2 & 6-\lambda & 2 \\ 0 & 2 & 5-\lambda \end{vmatrix}$$

$\boxed{\lambda_1 = 3}$

$\boxed{\lambda_2 = 6}$

$\boxed{\lambda_3 = 9}$

$$= (7-\lambda)(6-\lambda)(5-\lambda) - 4(5-\lambda) - 4(7-\lambda)$$
$$= (7-\lambda)(6-\lambda)(5-\lambda) - 4(12-2\lambda)$$
$$= (7-\lambda)(6-\lambda)(5-\lambda) - 8(6-\lambda)$$
$$= (6-\lambda)\left[(7-\lambda)(5-\lambda) - 8\right]$$
$$0 = (6-\lambda)(\lambda^2 - 12\lambda + 27) = (6-\lambda)(\lambda-3)(\lambda-9)$$

EV von A zu $\lambda = 3$:

$$A - 3E_3 = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

I: $4x_1 + 2x_2 = 0$
$\Leftrightarrow 2x_1 + x_2 = 0$
wähle $\boxed{x_1 = \alpha} \Rightarrow \boxed{x_2 = -2\alpha}$

III: $2x_2 + 2x_3 = 0 \Leftrightarrow x_2 + x_3 = 0 \Rightarrow \boxed{x_3 = 2\alpha}$

$$\Rightarrow Eig(A;3) = \left\{ \alpha \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$

EV von A zu $\lambda = 6$:

$$A - 6E_3 = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix}$$

III: $2x_2 - x_3 = 0$
wähle $\boxed{x_2 = \alpha} \Rightarrow \boxed{x_3 = 2\alpha}$

I: $x_1 + 2x_2 = 0 \Rightarrow \boxed{x_1 = -2\alpha}$

$$\Rightarrow Eig(A;6) = \left\{ \alpha \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$

EV von A zu $\lambda = 9$:

$$A - 9E_3 = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -4 \end{pmatrix}$$

III: $2x_2 - 4x_3 = 0 \Leftrightarrow x_2 - 2x_3 = 0$
wähle $\boxed{x_3 = \alpha} \Rightarrow \boxed{x_2 = 2\alpha}$

I: $-2x_1 + 2x_2 = 0 \Leftrightarrow x_1 = x_2 \Rightarrow \boxed{x_1 = 2\alpha}$

$$\Rightarrow Eig(A;9) = \left\{ \alpha \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$

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a) Wähle folgende EV:

zu $\lambda = 3$: $\vec{x}_1 = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \Rightarrow |\vec{x}_1| = 1$

zu $\lambda = 6$: $\vec{x}_2 = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \Rightarrow |\vec{x}_2| = 1$

zu $\lambda = 9$: $\vec{x}_3 = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \Rightarrow |\vec{x}_3| = 1$

b) $S = (\vec{x}_1, \vec{x}_2, \vec{x}_3)$

$$S = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \Longrightarrow S^t = S$$

da S symmetrisch

$$\vec{x}_1 \cdot \vec{x}_2 = \frac{1}{9}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{9}(-2 -2 +4) = 0 \Longrightarrow \vec{x}_1 \perp \vec{x}_2$$

$$\vec{x}_1 \circ \vec{x}_3 = \frac{1}{9}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \circ \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{9}(2 -4 +2) = 0 \Longrightarrow \vec{x}_1 \perp \vec{x}_3$$

$$\vec{x}_2 \cdot \vec{x}_3 = \frac{1}{9}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \circ \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{9}(-4 +2 +2) = 0 \Longrightarrow \vec{x}_2 \perp \vec{x}_3$$

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c)
$$SS^t = \frac{1}{9}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Longrightarrow S^t = S^{-1}$$

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d)
$$AS = \frac{1}{3}\begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 6 \\ -2 & 2 & 6 \\ 2 & 4 & 3 \end{pmatrix}$$

$$S^tAS = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -4 & 6 \\ -2 & 2 & 6 \\ 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix} = D$$