Lösungen

Afg 1

a) $y = \sin x - x\cos x$

$y' = \cos x - (1\cdot\cos x - x\cdot\sin x) = \boxed{x\sin x}$

b) $y = (x^2-3)\sin x + 2x\cos x$

$y' = 2x\sin x + (x^2-3)\cos x + 2\cos x - 2x\sin x$

$= (x^2-3+2)\cos x = (x^2-1)\cos x$

c) $y = \sqrt{e^x}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)$

$y' = \frac{1}{2}\sqrt{e^x}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) + \sqrt{e^x}\left(\frac{1}{2}\cos\frac{x}{2} - \frac{1}{2}\sin\frac{x}{2}\right)$

$= \frac{1}{2}\sqrt{e^x}\left(\sin\frac{x}{2} + \cos\frac{x}{2} + \cos\frac{x}{2} - \sin\frac{x}{2}\right)$

$= \frac{1}{2}\sqrt{e^x}\left(2\cos\frac{x}{2}\right) = \sqrt{e^x}\cos\frac{x}{2}$

d) $y = 3\sin(2x) + (\pi - 6x)\cos(2x)$

$y' = 6\cos(2x) - 6\cos(2x) - (\pi-6x)\cdot 2\sin(2x)$

$= 2(6x-\pi)\sin(2x)$

e) $y = \dfrac{x^2+3}{(1-3x)^2}$

$y' = \dfrac{2x(1-3x)^2 - 2(1-3x)(-3)(x^2+3)}{(1-3x)^4}$

$= \dfrac{2x(1-3x) + 6(x^2+3)}{(1-3x)^3} = \dfrac{2x+18}{(1-3x)^3}$

zu Afg 1:

f) $$y = \sqrt{\frac{2x+1}{1-x}}$$

$$y' = \frac{1}{2}\sqrt{\frac{1-x}{2x+1}} \cdot \frac{2(1-x)+2x+1}{(1-x)^2}$$

$$= \frac{1}{2}\sqrt{\frac{1-x}{2x+1}} \cdot \frac{2-2x+2x+1}{(1-x)^2}$$

$$= \frac{3}{2(1-x)^2}\sqrt{\frac{1-x}{2x+1}}$$

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g) $$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$y' = \frac{1}{2} \cdot \frac{1-x}{1+x} \cdot \frac{1-x+1+x}{(1-x)^2} = \frac{1}{2} \cdot \frac{1-x}{1+x} \cdot \frac{2}{(1-x)^2}$$

$$= \frac{1}{(1+x)(1-x)} = \frac{1}{1-x^2}$$

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h) $$y = \frac{2\cos x - \sin x}{\cos x + 2\sin x}$$

$$y' = \frac{(-2\sin x - \cos x)(\cos x + 2\sin x) - (2\cos x - \sin x)(-\sin x + 2\cos x)}{(\cos x + 2\sin x)^2}$$

$$= \frac{-(2\sin x + \cos x)^2 - (2\cos x - \sin x)^2}{(\cos x + 2\sin x)^2}$$

$$= \frac{-4\sin^2 x - \cancel{4\sin x\cos x} - \cos^2 x - 4\cos^2 x + \cancel{4\sin x\cos x} - \sin^2 x}{(\cos x + 2\sin x)^2}$$

$$= \frac{-5\sin^2 x - 5\cos^2 x}{(\cos x + 2\sin x)^2} = -\frac{5(\sin^2 x + \cos^2 x)}{(\cos x + 2\sin x)^2}$$

$$= -\frac{5}{(\cos x + 2\sin x)^2}$$; i) und j) siehe S. 4!

zu Afg 2, h) $\quad x_0 = 4$

$$h(x) = 2x-6+\frac{1}{(x-3)^2} \; ; \; h'(x) = 2-\frac{2}{(x-3)^2} \; ; \; h''(x) = \frac{6}{(x-3)^4}$$

$$y = h(4) = 3 \; ; \; y' = h'(4) = 0 \; ; \; y'' = h''(4) = 6$$

$$a = x_0 - \frac{y'(1+(y')^2)}{y''} = 4 - \frac{0\cdot(\ldots)}{6} = 4$$

$$b = \frac{1+(y')^2}{y''} + y = \frac{1+0}{6} + 3 = \frac{1}{6} + \frac{18}{6} = \frac{19}{6}$$

$$\Rightarrow \boxed{M\left(4 / \tfrac{19}{6}\right)}$$

$$r = \frac{(1+(y')^2)^{1,5}}{|y''|} = \frac{(1+0)^{1,5}}{6} = \frac{1}{6}$$

Afg 6 $\quad v(t) = -7\left(\frac{t}{5}-1\right)^4 + 7$

a) gem. Formel $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$ mit $F' = f$

gilt $(a = \frac{1}{5}\; ; \; b = -1, \; f(z) = z^4)$

$$\int v(t)\,dt = -7\,\frac{\left(\frac{t}{5}-1\right)^5}{5}\cdot 5 + 7t + C$$

$$s(t) = -7\left(\frac{t}{5}-1\right)^5 + 7t + C$$

Anfangsbed. $s(0) = 0 \Rightarrow -7(-1)^5 + C = 0$

$$\Leftrightarrow 7 + C = 0 \Leftrightarrow C = -7$$

$$\Rightarrow \boxed{s(t) = -7\left(\frac{t}{5}-1\right)^5 + 7t - 7}$$

zu Afg 5

b) zurückgelegter Weg im Zeitintervall [1s ; 2s]:

$$\int_1^2 v(t)\,dt = s(2) - s(1)$$
$$= 7{,}544 - 2{,}294$$
$$= \boxed{5{,}25\ m}$$

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zu Afg 1

i) $$y = \frac{\ln x}{\sqrt{x}}$$

$$y' = \frac{\frac{1}{x}\cdot\sqrt{x} - \frac{1}{2\sqrt{x}}\cdot\ln x}{x} \quad \begin{matrix}\cdot 2\sqrt{x}\\ \cdot 2\sqrt{x}\end{matrix}$$

$$= \frac{\frac{1}{x}\cdot 2x - \ln x}{2x\sqrt{x}} = \frac{2-\ln x}{2x\cdot x^{1/2}}$$

$$= \frac{2-\ln x}{2\cdot x^{3/2}} = \frac{2-\ln x}{2\sqrt{x^3}}$$

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j) $$y = e^{-x^2}\ln(x^3+1)$$

$$y' = -2xe^{-x^2}\ln(x^3+1) + e^{-x^2}\cdot\frac{3x^2}{x^3+1}$$
$$= xe^{-x^2}\left(\frac{3x}{x^3+1} - 2\ln(x^3+1)\right)$$

Afg 3

Mit Newton-Verfahren Näherungslösung x_n für die Gleichung

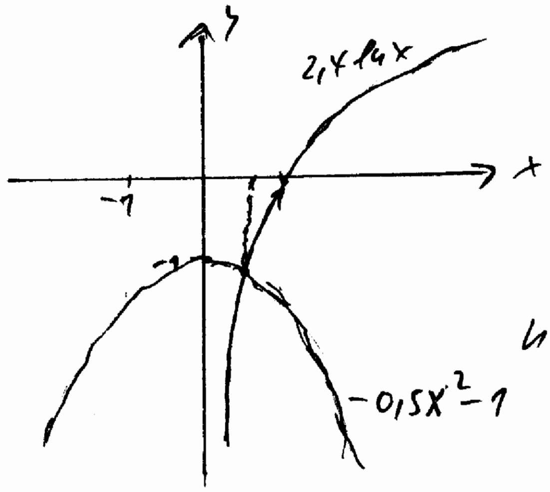
$$f(x) = 2{,}4 \ln x + 0{,}5x^2 + 1 = 0$$

berechnen, so dass $f(x_n) = 0{,}0000\ldots$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

zunächst geeigneten Startwert x_0 finden.
Mache zur Orientierung eine Skizze:

$$2{,}4 \ln x + 0{,}5x^2 + 1 = 0 \iff 2{,}4 \ln x = -0{,}5x^2 - 1$$



$f(1) = 1{,}5 > 0$

$f(0{,}5) = -0{,}539 < 0$

nehme $x_0 = \frac{1{,}5}{2} = 0{,}75$

n	x_n	$f(x_n)$
0	0,75	0,59081
1	0,60043	−0,044
2	0,61000	−0,00026
3	$\boxed{0{,}61006}$	0,00001

$\boxed{x_3 = 0{,}61006}$

geeignete Näherungslösung!

zu Afg 4 (6)

m) $\int a^2 t e^{-\frac{a}{b}t} dt \overset{\text{part.}}{\underset{\text{Int.}}{=}} a^2 t \cdot (-\frac{b}{a}) e^{-\frac{a}{b}t} - \int a^2 (-\frac{b}{a}) e^{-\frac{a}{b}t} dt$

$= -abt e^{-\frac{a}{b}t} + ab \int e^{-\frac{a}{b}t} dt$

$= -abt e^{-\frac{a}{b}t} + ab \cdot (-\frac{b}{a}) e^{-\frac{a}{b}t}$

$= -abt e^{-\frac{a}{b}t} - b^2 e^{-\frac{a}{b}t}$

$= -b e^{-\frac{a}{b}t}(at+b) + C$

n) $\int \sqrt{4x-7}\, dx = \int (4x-7)^{\frac{1}{2}} dx$

$= \frac{1}{4} \cdot \frac{(4x-7)^{1/2+1}}{1/2+1} = \frac{1}{4} \frac{(4x-7)^{3/2}}{3/2}$

$= \frac{1}{4} \cdot \frac{2}{3} (4x-7)^{3/2} = \frac{1}{6} \sqrt{(4x-7)^3}$

Beachte allgemein: $\int f(ax+b) dx = \frac{1}{a} F(ax+b)$

mit $F' = f$

Jetzt

$\int_2^3 \sqrt{4x-7}\, dx = \frac{1}{6} \left[\sqrt{(4x-7)^3} \right]_2^3$

$= \frac{1}{6} (\sqrt{5^3} - \sqrt{1}) = 1{,}6967$

$= \frac{1}{6} (\sqrt{125} - 1)$

o) $\int e^{-\frac{mgh}{RT}} dh$

$= -\frac{RT}{mg} \cdot e^{-\frac{mgh}{RT}}$

Afg 7

Berechne numerisch $\int_0^{0,5} \cos(\sqrt{x})\,dx$

auf 5 Nachkommastellen genau. Zerlege das Int.-intervall in 5 Teilintervalle:

Es gilt $\int_a^b f(x)\,dx \approx \left(\frac{y_0+y_n}{2} + \sum_{i=1}^{n-1} y_i\right)\cdot \frac{b-a}{n}$

gem. Trapezformel. Es ist $y_i = f\left(a + i\cdot\frac{b-a}{n}\right)$
$i = 0;1;2;\dots;n$

$f(x) = \cos(\sqrt{x})$, $a = 0$; $b = 0,5$; $n = 5$

$y_i = f\left(i\cdot\frac{0,5}{5}\right) = \cos(\sqrt{0,1i})$; $i = 0;1;2;\dots;5$

TR auf Bogenmaß umstellen!
[RAD]

i	y_i
0	$\cos(0) = 1$
1	$\cos(\sqrt{0,1}) = 0,950415$
2	$\cos(\sqrt{0,2}) = 0,901656$
3	$\cos(\sqrt{0,3}) = 0,853713$
4	$\cos(\sqrt{0,4}) = 0,806578$
5	$\cos(\sqrt{0,5}) = 0,760245$

$\sum_{i=1}^{4} y_i = 3,512362$

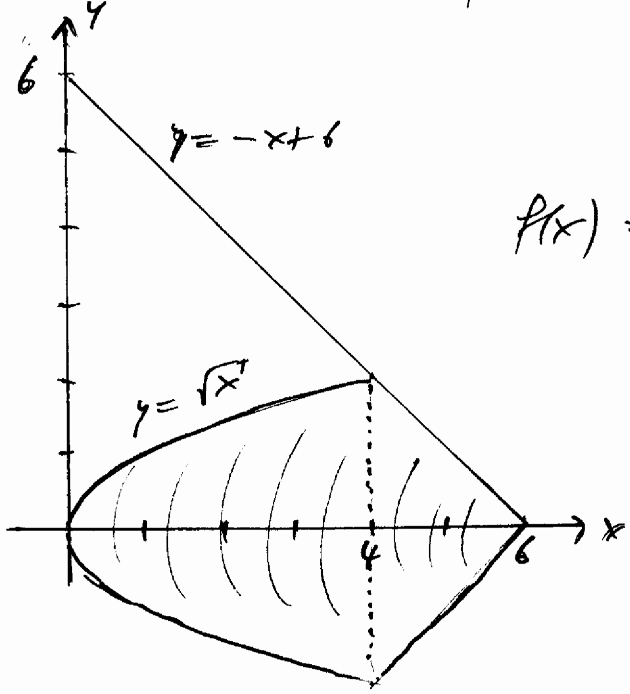
$$\Rightarrow \int_0^{0,5}\cos(\sqrt{x})\,dx \approx \left(\frac{y_0+y_5}{2} + \sum_{i=1}^{4} y_i\right)\cdot 0,1$$

$$= \left(\frac{1,760245}{2} + 3,512362\right)\cdot 0,1$$

$$= \boxed{0,43925}\ \text{FE}$$

Afg 8

$y = \sqrt{x}$ und $y = -x + 6$



$$f(x) = \begin{cases} \sqrt{x} & \text{für } 0 \le x \le 4 \\ -x+6 & \text{für } 4 \le x \le 6 \end{cases}$$

$$\text{Volumen } V = \pi \int_0^6 f^2(x)\,dx$$

$$= \pi \left(\int_0^4 x\,dx + \int_4^6 (x-6)^2\,dx \right)$$

$$= \pi \left(\left[\frac{x^2}{2} \right]_0^4 + \left[\frac{(x-6)^3}{3} \right]_4^6 \right)$$

$$= \pi \left(\frac{16}{2} + \frac{8}{3} \right) = \frac{32\pi}{3} \approx \boxed{33{,}51 \text{ FE}}$$

Afg 9

a) $1 + \frac{2}{3} + \frac{3}{9} + \frac{4}{27} + \frac{5}{81} + \dots = \sum_{n=1}^{\infty} \frac{n}{3^{n-1}}$

Wende Quotientenkriterium an: $a_n = \frac{n}{3^{n-1}}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{3^n}}{\frac{n}{3^{n-1}}} = \frac{n+1}{3^n} \cdot \frac{3^{n-1}}{n}$$

$$= \frac{n+1}{3 \cdot \cancel{3^{n-1}}} \cdot \frac{\cancel{3^{n-1}}}{n} = \frac{n+1}{3n} = \frac{1}{3} + \frac{1}{3n}$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{3n} = \frac{1}{3} < 1$$

$\Rightarrow$ Reihe konvergiert!

b) $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

Die Folge $a_n = \frac{1}{n^2}$ ist smf und $\lim_{n\to\infty} a_n = 0$

nach Leibnizkriterium $\Rightarrow$ Reihe konvergiert!

c) $P(x) = 1 + \frac{2x}{2} + \frac{3x^2}{4} + \frac{4x^3}{8} + \frac{5x^4}{16} + \dots = \sum_{n=0}^{\infty} \frac{n+1}{2^n} \cdot x^n$

Bestimme Konvergenzradius r $\qquad a_n = \frac{n+1}{2^n}$

$$\frac{a_n}{a_{n+1}} = \frac{n+1}{2^n} \cdot \frac{2^{n+1}}{n+2} = \frac{n+1}{\cancel{2^n}} \cdot \frac{2 \cdot \cancel{2^n}}{n+2} = \frac{2(n+1)}{n+2}$$

$\Rightarrow r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = 2 \Rightarrow$ konvergiert für $|x| < 2$

Prüfe, ob Reihe für $|x| = 2$ konvergiert:

$x = 2$: $\sum_{n=0}^{\infty} \frac{n+1}{2^n} \cdot 2^n = \sum_{n=0}^{\infty} (n+1) = \infty$ divergiert

und $x = -2$: $\sum_{n=0}^{\infty} \frac{n+1}{2^n} \cdot (-2)^n = \sum_{n=0}^{\infty} (-1)^n (n+1)$ divergiert

$\Rightarrow$ Konv.bereich $]-2; 2[$.

Afg 10

$$f(x) = \sqrt{1+x^2} = (1+x^2)^{1/2}$$

$$f(x) = \sum_{n=0}^{\infty} \binom{1/2}{n} (x^2)^n = \sum_{n=0}^{\infty} \binom{1/2}{n} x^{2n}$$

$$\binom{1/2}{0} = 1 \quad ; \quad \binom{1/2}{1} = 1/2$$

$$\binom{1/2}{2} = \frac{1/2 \cdot (1/2 - 1)}{2!} = \frac{1}{2} \cdot \frac{1}{2} \left(-\frac{1}{2}\right) = -\frac{1}{8}$$

$$\binom{1/2}{3} = \frac{1/2\,(1/2 - 1)(1/2 - 2)}{3!} = \frac{1}{6} \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) = \frac{1}{16}$$

beachte: allg. $\boxed{\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots\cdots(\alpha-n+1)}{n!}}$

daraus folgt

$$f(x) = \binom{1/2}{0} + \binom{1/2}{1} x^2 + \binom{1/2}{2} x^4 + \binom{1/2}{3} x^6 + \cdots$$

$$= \underbrace{\underbrace{\underbrace{1 + \frac{1}{2} x^2}_{1.\ \text{Näherung} = f_1(x)} - \frac{1}{8} x^4}_{2.\ \text{Näherung} = f_2(x)} + \frac{1}{16} x^6}_{3.\ \text{Näherung} = f_3(x)} - \cdots$$

$$f(0{,}5) = \sqrt{1 + 0{,}5^2} = 1{,}118034$$

Für die Näherungen ergibt sich

i	$f_i(0{,}5)$	%-Abw. $= \frac{\lvert f_i(0{,}5) - f(0{,}5)\rvert}{f(0{,}5)}$
1	1,125000	0,62 %
2	1,117188	0,08 %
3	1,118164	0,01 %